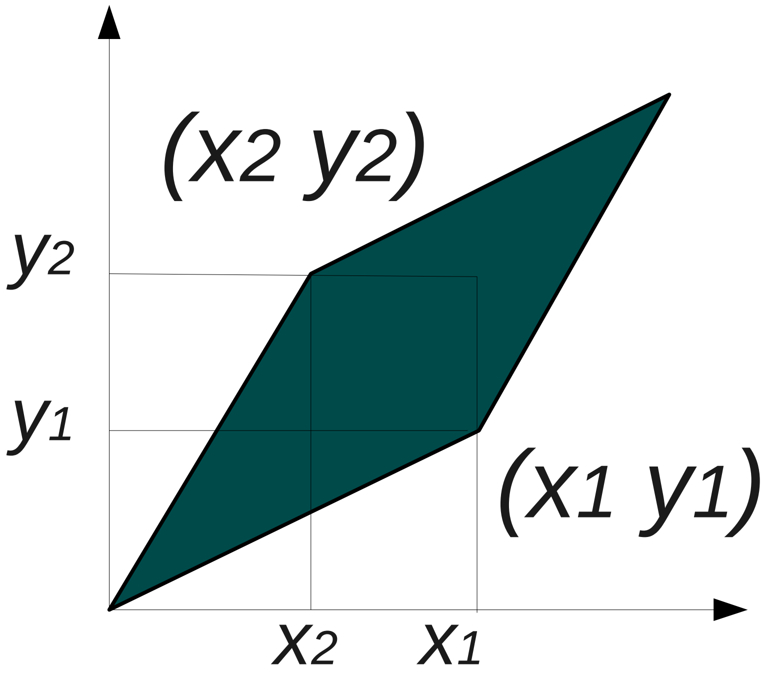
Why parallelogram area is $|x_1 y_2 - x_2 y_1|$?
(and it's 2x2 matrix determinant.)

2011-07-10

Yamauchi, Hitoshi
Sunday Researcher

Why parallelogram area is $|x_1 y_2 - x_2 y_1|$?

- My question:



The area of parallelogram:

$$\left|\det\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}\right| = |x_1 y_2 - x_2 y_1|$$

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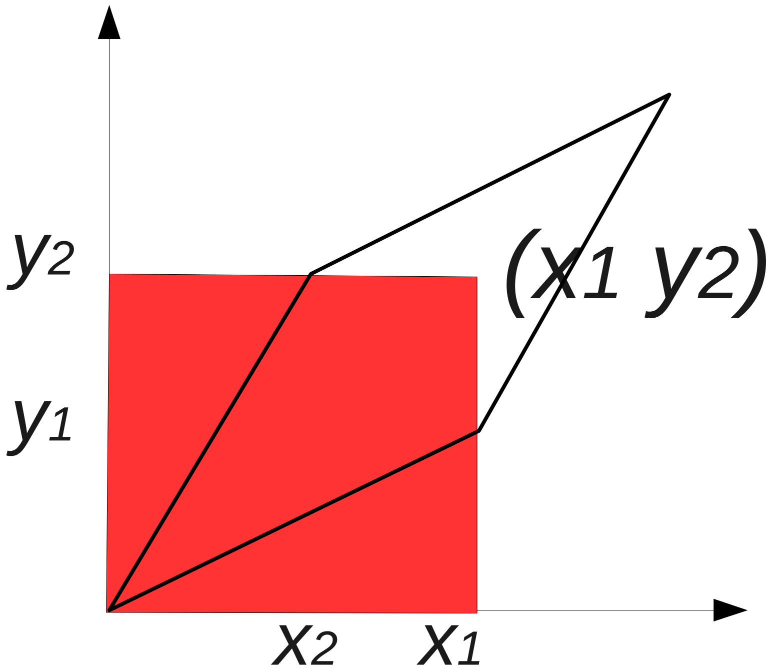

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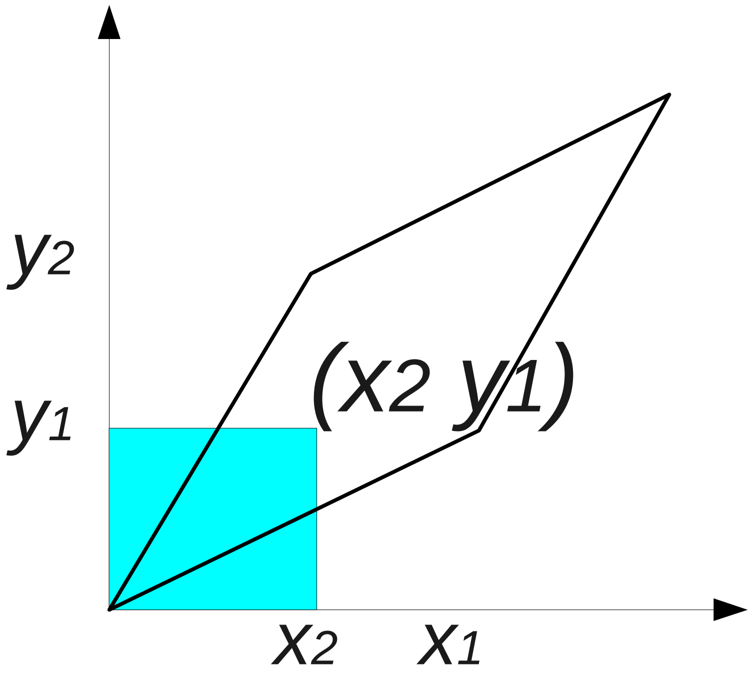

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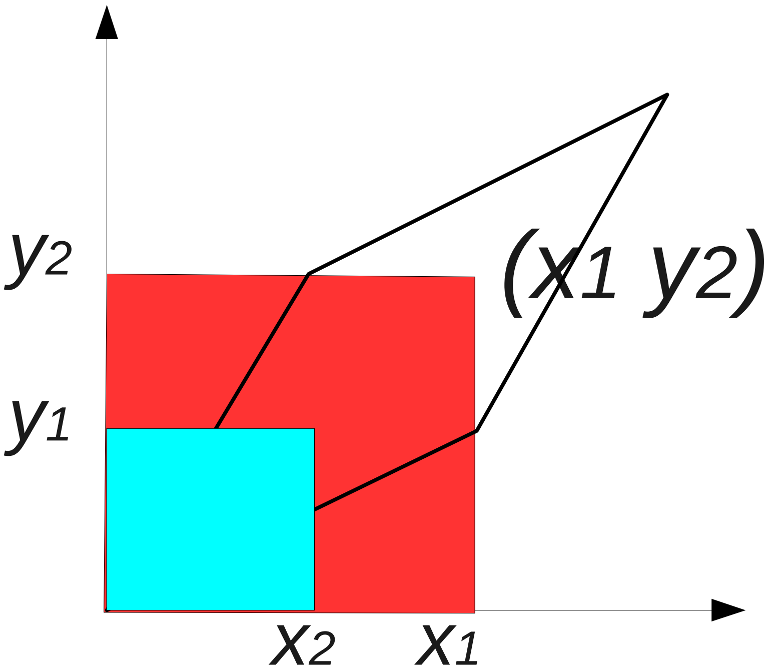

The area of parallelogram:

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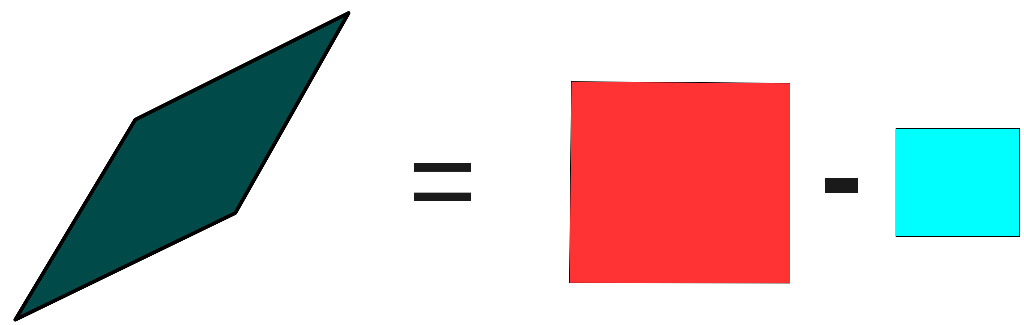
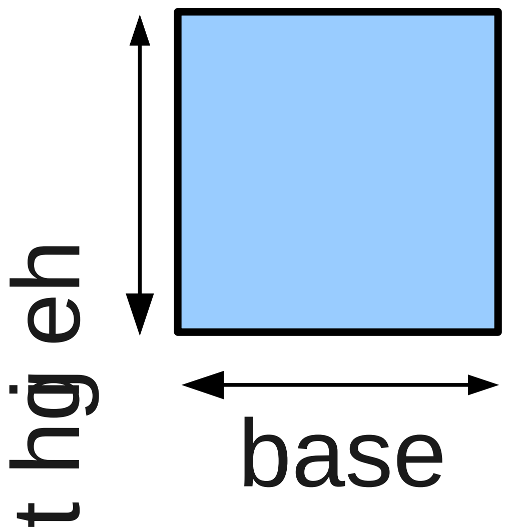


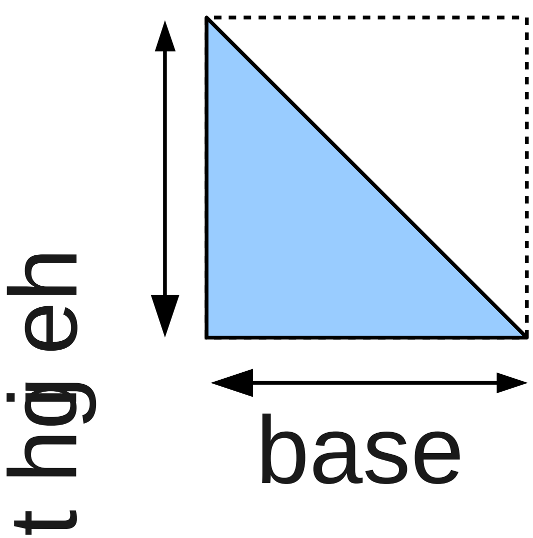
Why is this? is my question.

Back to basics:
the area of rectangle and triangle



- Area of rectangle

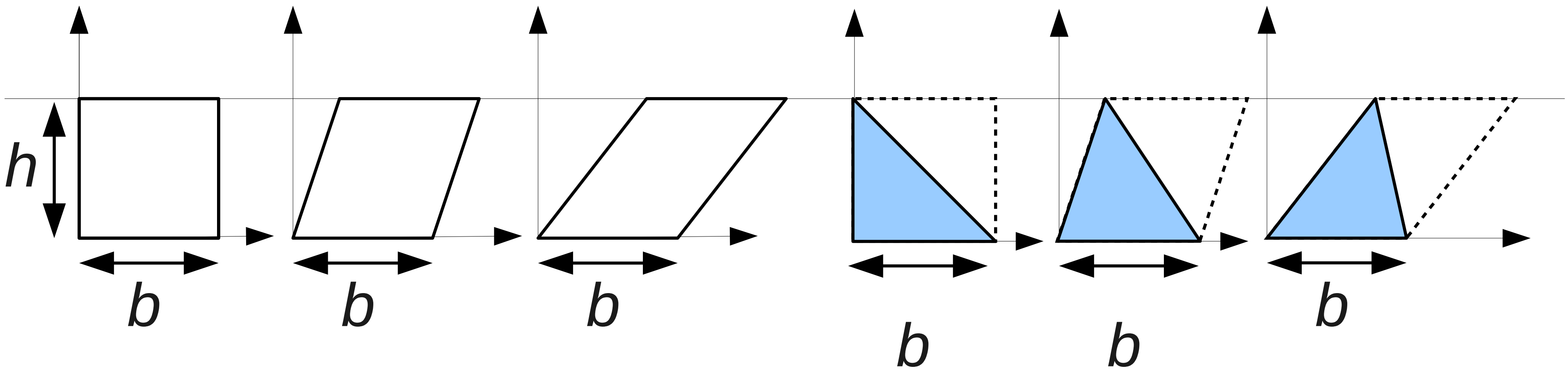
  = base * height



- Area of triangle

  = ½ base * height

Shear doesn't change the area

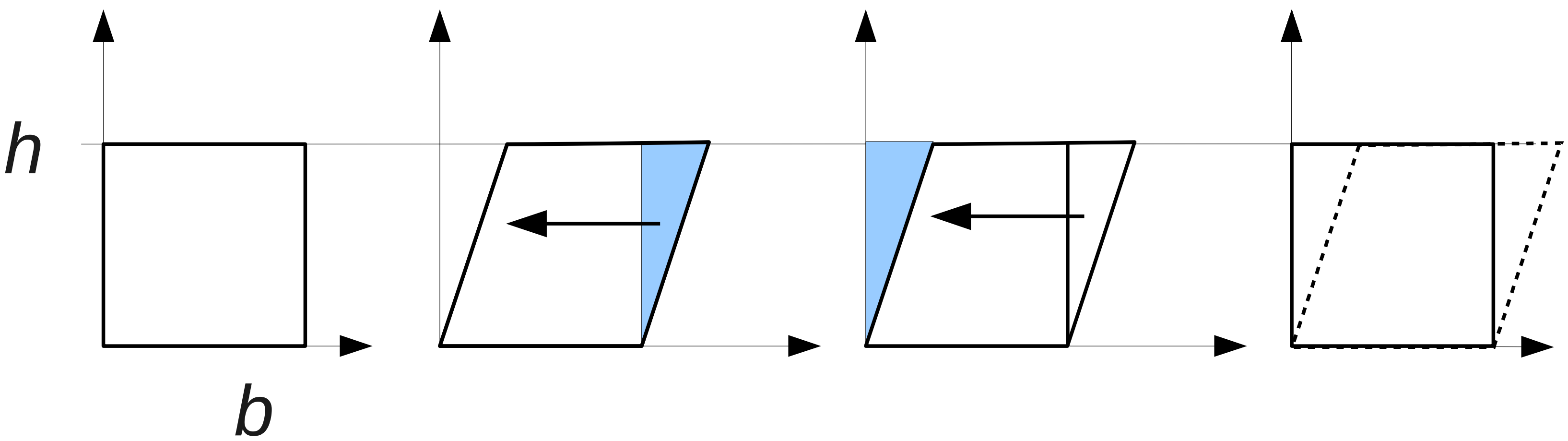


- Shearing rectangle/triangle: no area change
- Because the base and height don't change

Area of rectangle = b * h

Area of triangle = ½ b * h

Shear doesn't change the area



Cut the blue triangle

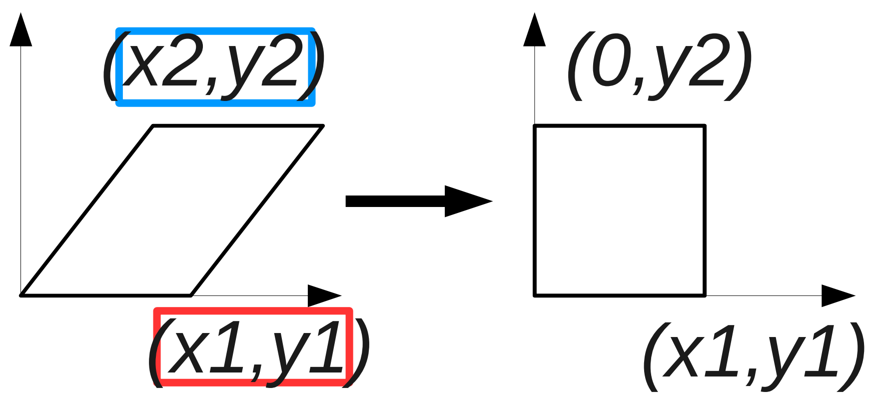
Paste to the left

- Another explanation:
  - Cut the blue triangle and paste to the left
  - The same area of b * h

Shear and Gaussian elimination

- Relationship between linear algebra
  - Gaussian elimination is shearing.

$$\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}$$
$$\rightarrow \begin{pmatrix} x_1 & y_1 \\ x_2 - \frac{x_2}{x_1}x_1 & y_2 - \frac{x_2}{x_1}y_1 \end{pmatrix}$$
$$\rightarrow \begin{pmatrix} x_1 & y_1 \\ 0 & y_2 \end{pmatrix}$$

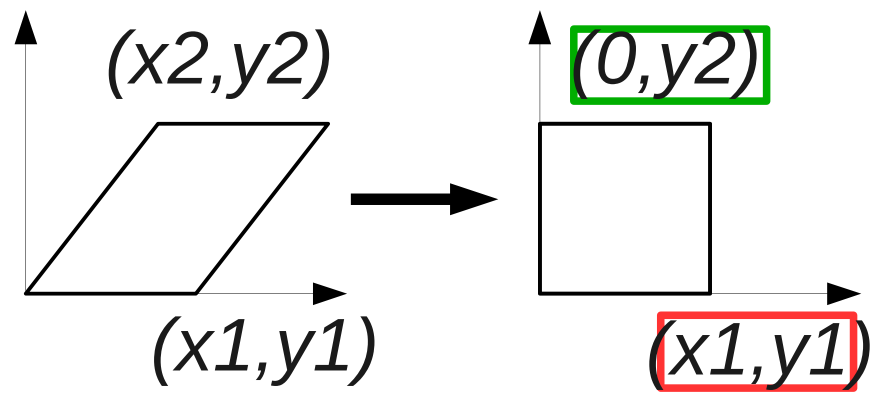


- row2 → row2 – row1 * x2/x1
- (The last '→' is because y1 is 0 in this particular case, usually y2 also changes)

Shear and Gaussian elimination

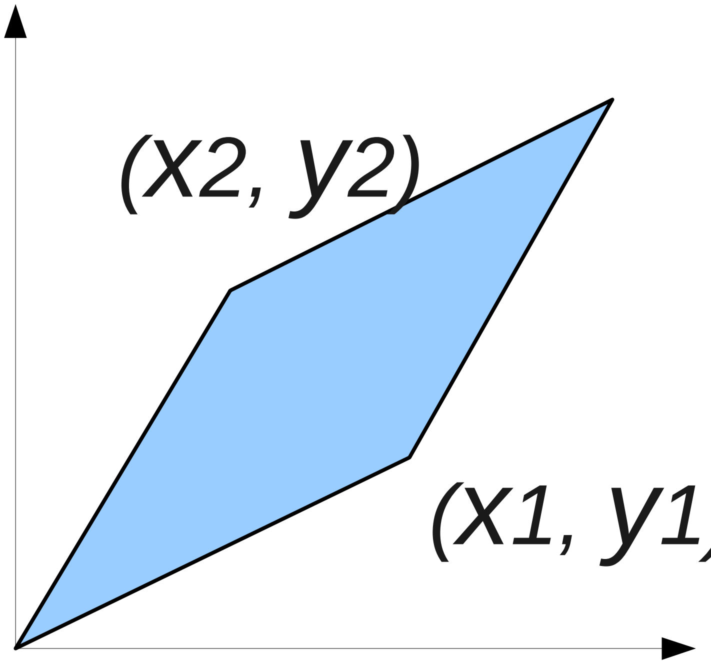
- Relationship between linear algebra
  - Gaussian elimination is shearing.

$$
\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}
\rightarrow \begin{pmatrix} x_1 & y_1 \\ x_2 - \frac{x_2}{x_1}x_1 & y_2 - \frac{x_2}{x_1}y_1 \end{pmatrix}
\rightarrow \begin{pmatrix} x_1 & y_1 \\ 0 & y_2 \end{pmatrix}
$$



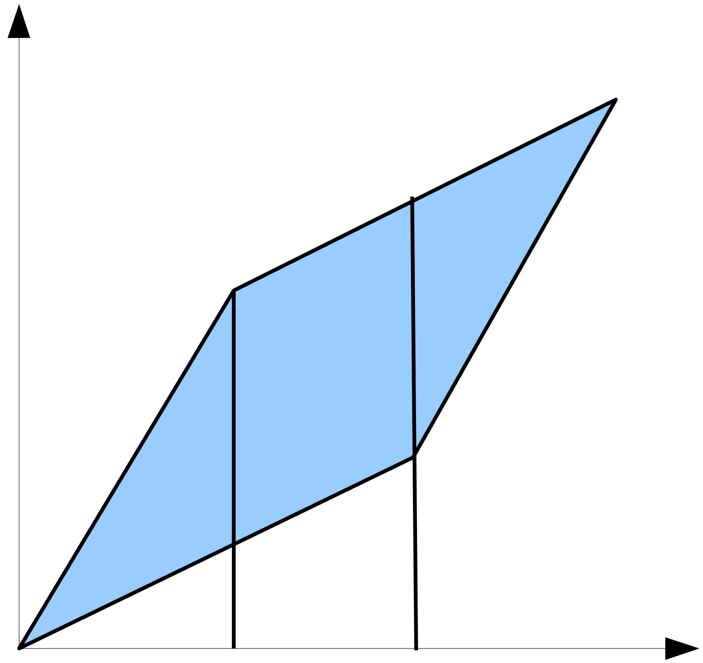
- Gaussian elimination doesn't change the area
  - area == determinant
- Determinant doesn't change!

Parallelogram area



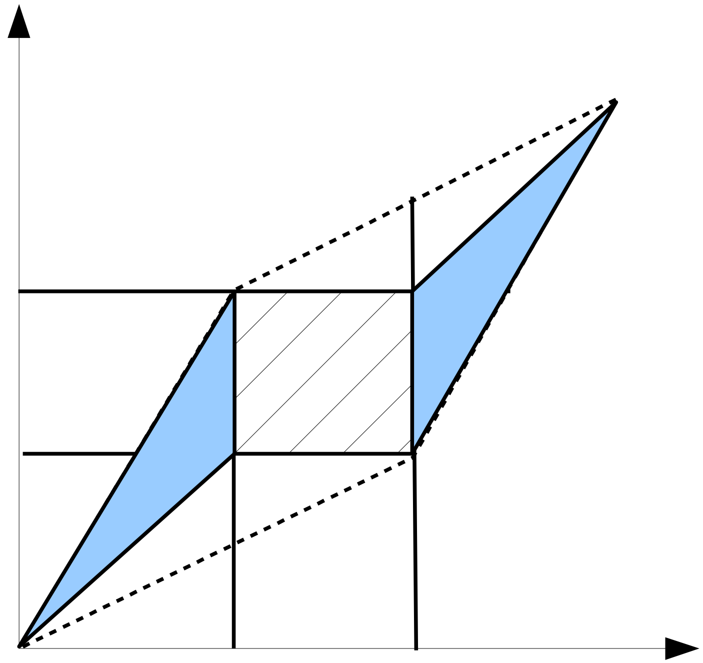
- Let's start with this figure

Parallelogram area



- Cut the parallelogram in three parts

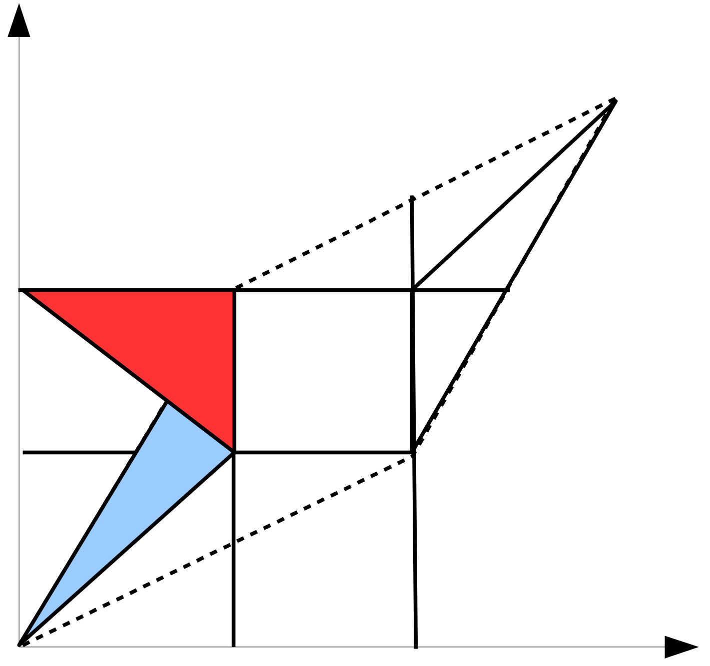
Parallelogram area

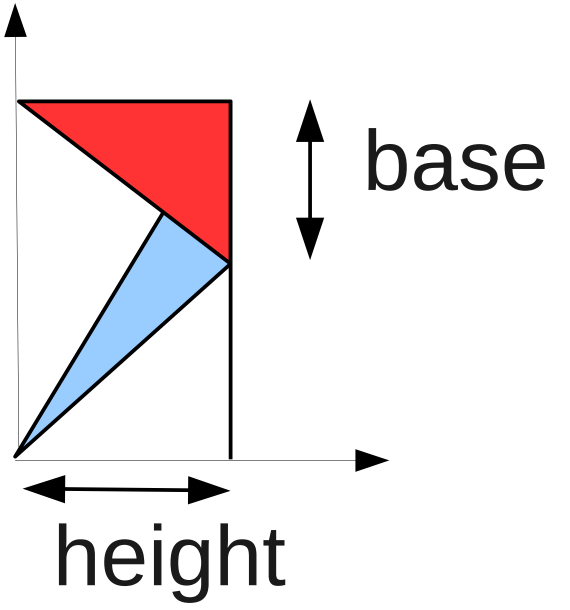


- More cut
- Think these two triangles
- Their area are the same
  - ½ base * height are the same

Parallelogram area

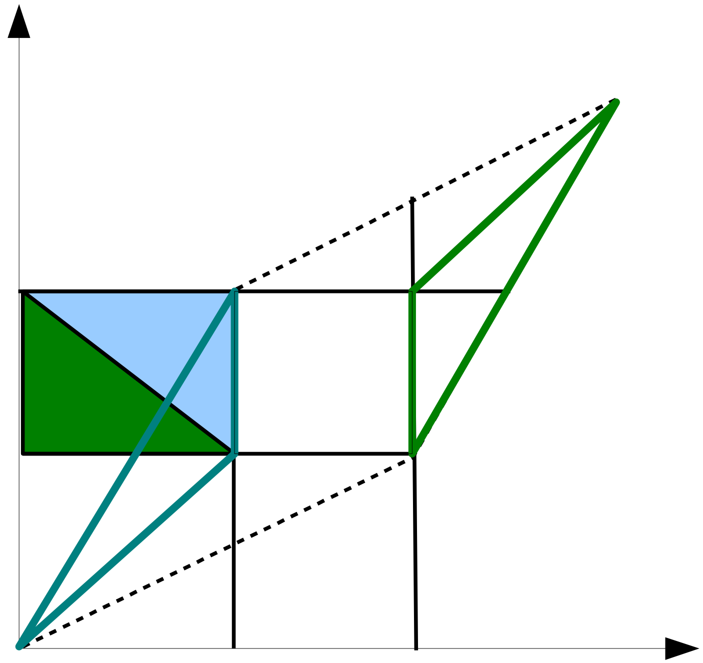
- Blue and red triangles have the same area
  - ½ base * height

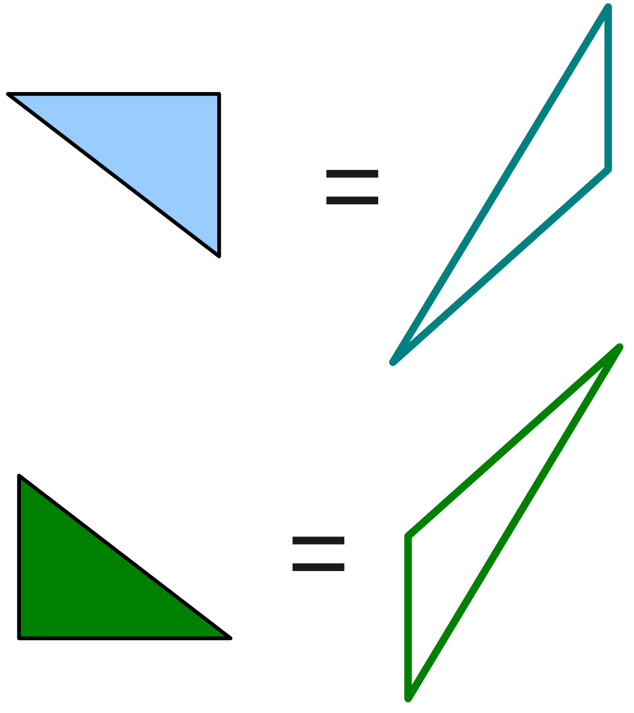




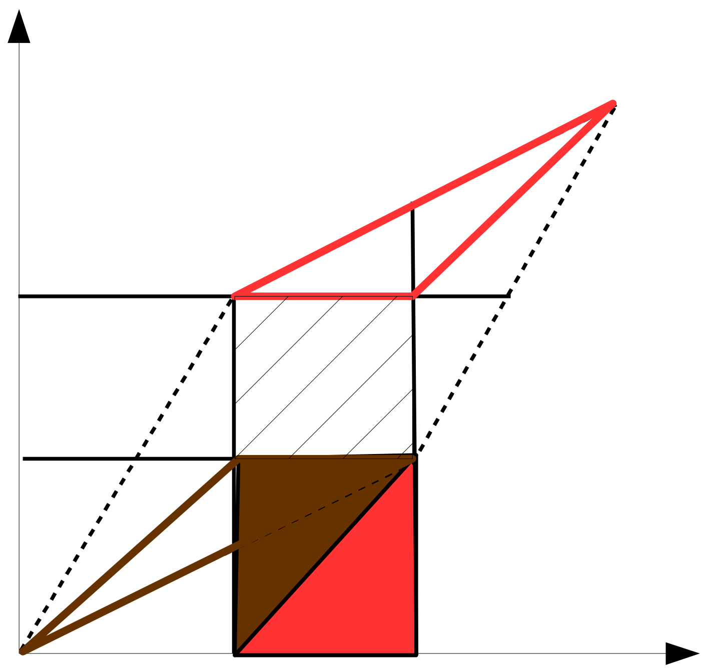
Parallelogram area

- These are the same area

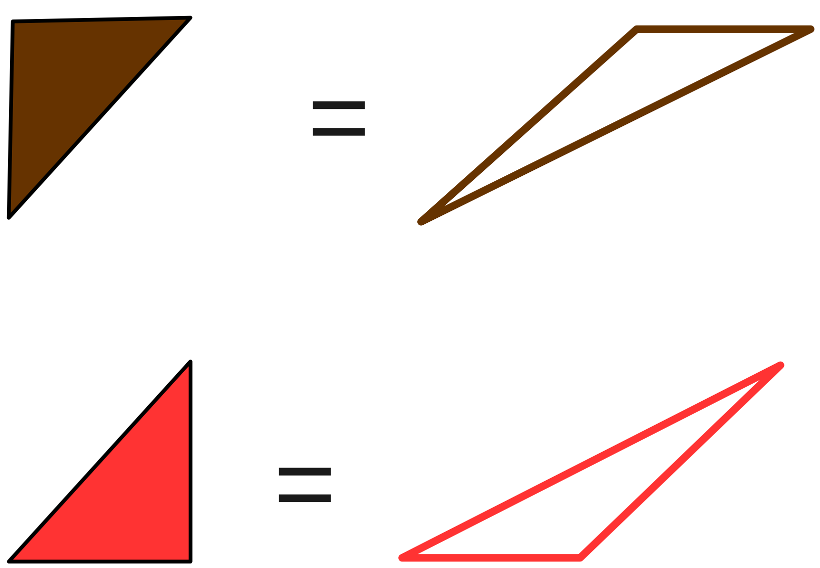




Parallelogram area

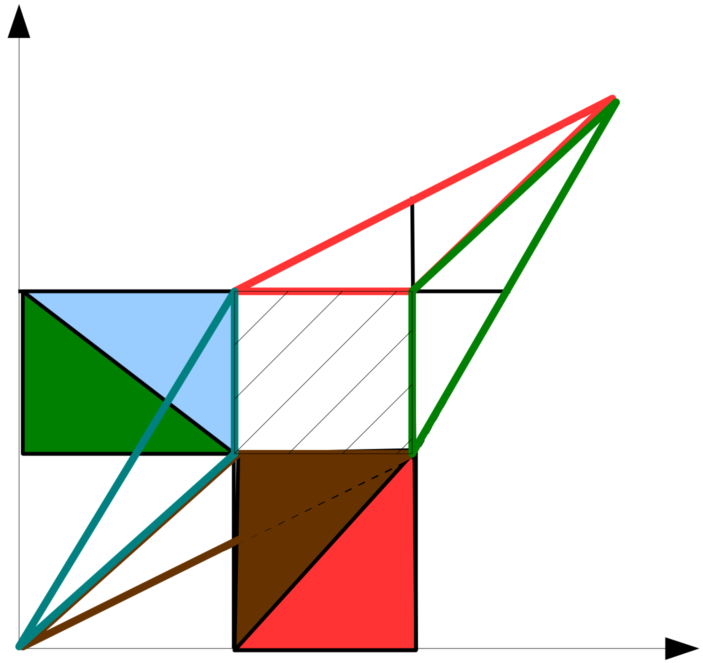


- In the same way, these are the same area

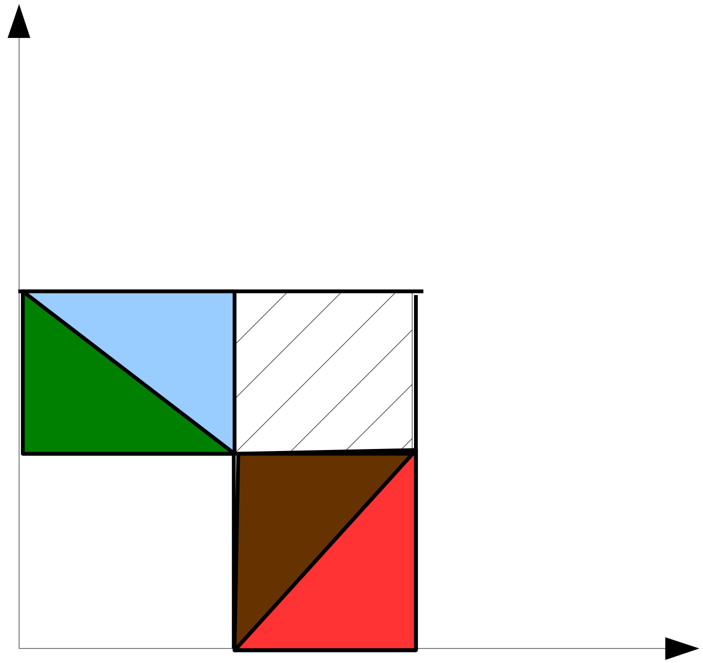


Parallelogram area

- All together



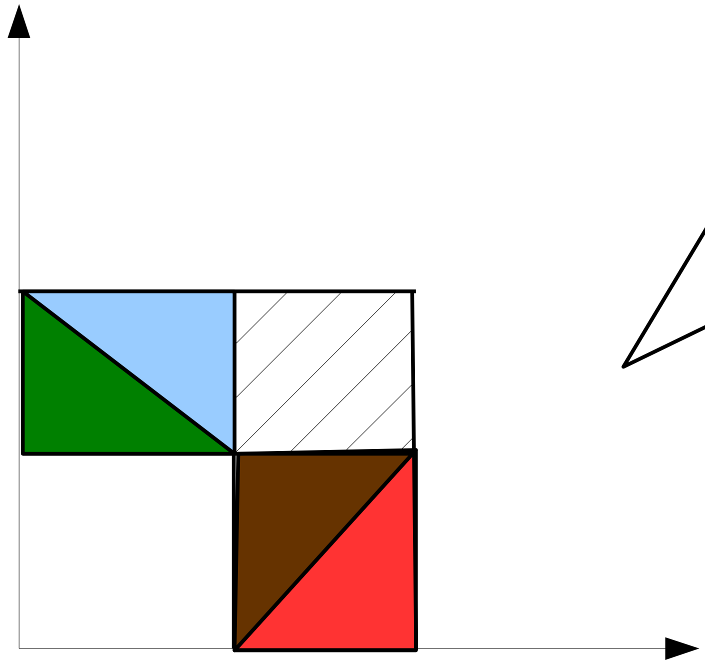
Parallelogram area

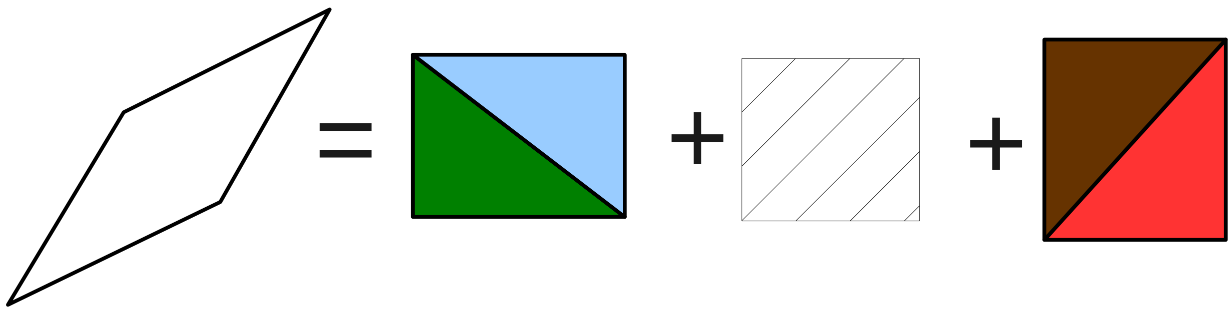


- All together
  - This is the parallelogram area

Parallelogram area

- Breakdown the area





Parallelogram area

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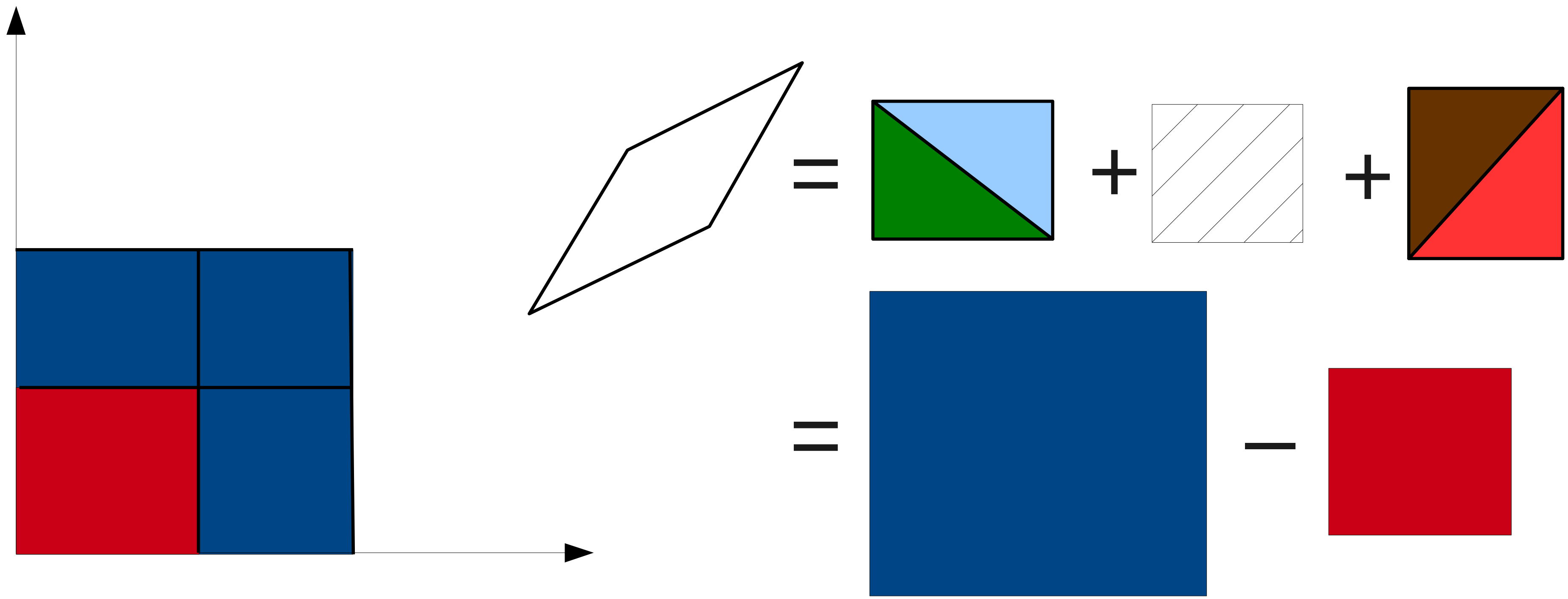
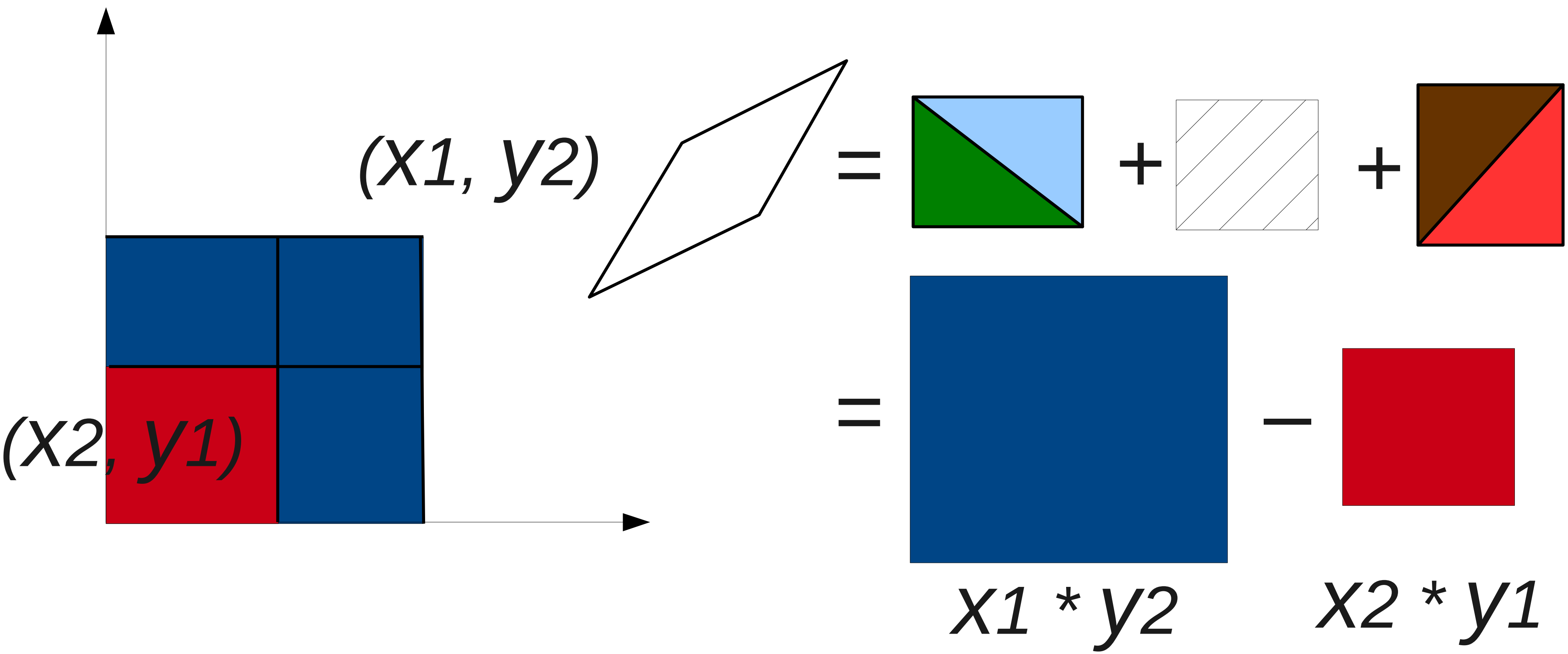


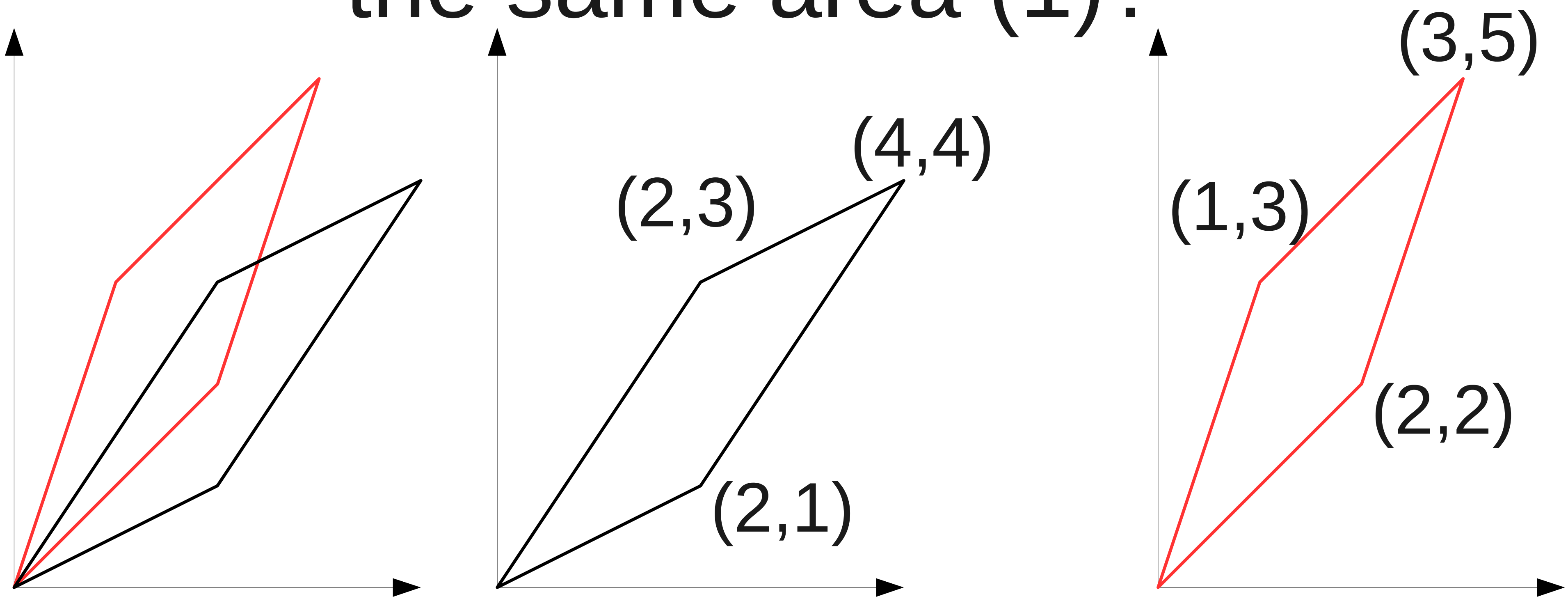
Parallelogram area

- Put the coordinates



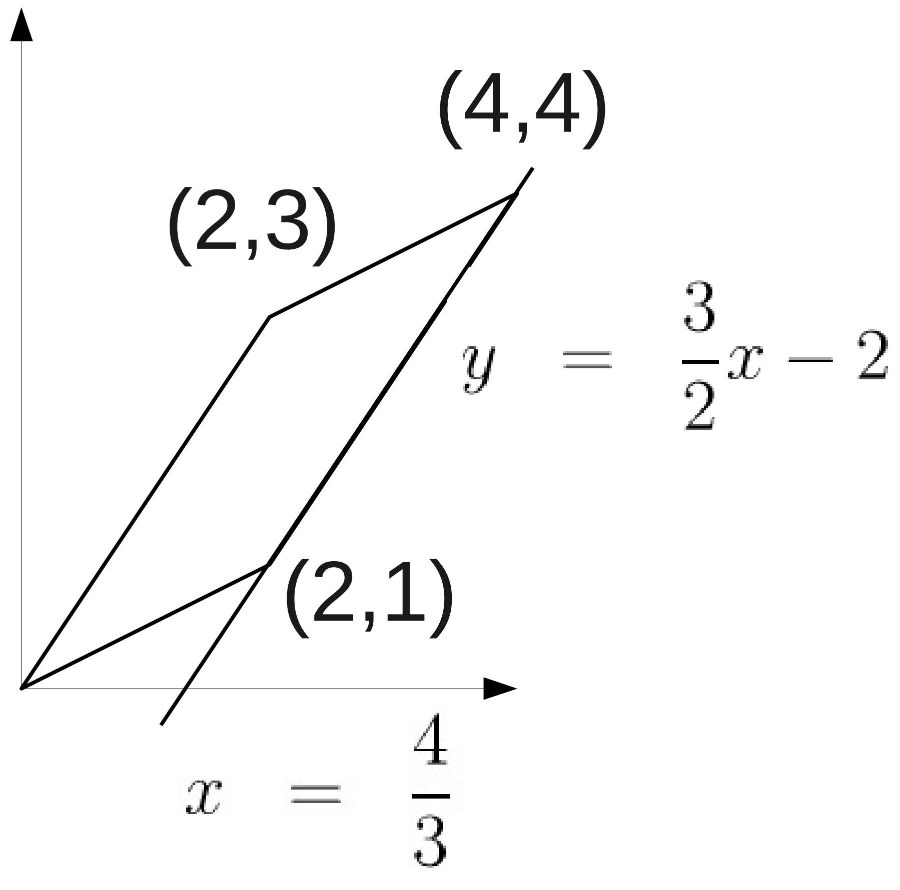
$$= |x_1 y_2 - x_2 y_1|$$

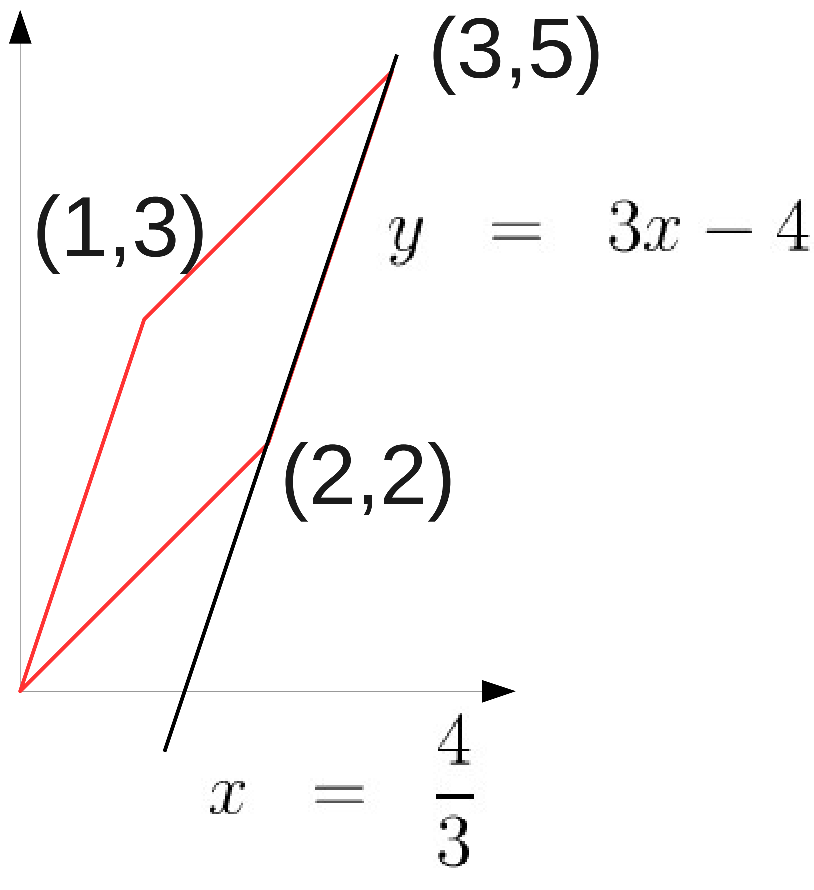
Why these two parallelogram has the same area (1)?



- These parallelograms have the same area
- But hard to see from the picture...
- [1] Gilbert Strang, Introduction to linear algebra, 4th ed. Chapter 5.3, Question 19

Why these two parallelogram has the same area (1)?





- The line equation tells us y=0 points are the same

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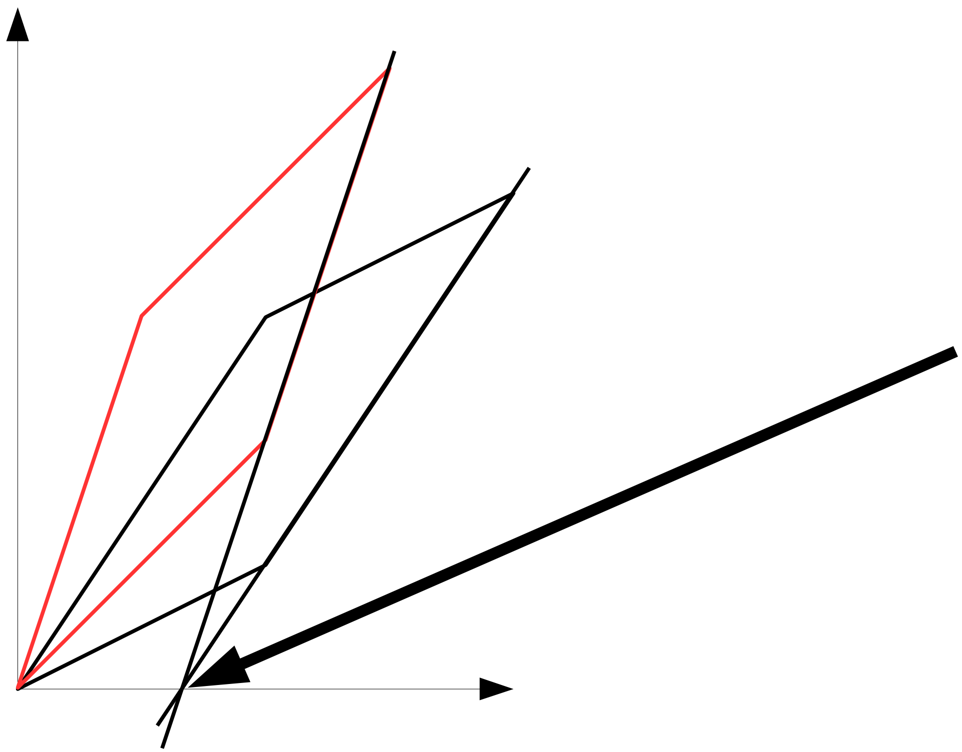

$$x = \frac{4}{3}$$

y = 0

- The line equation tells us y=0 points are the same

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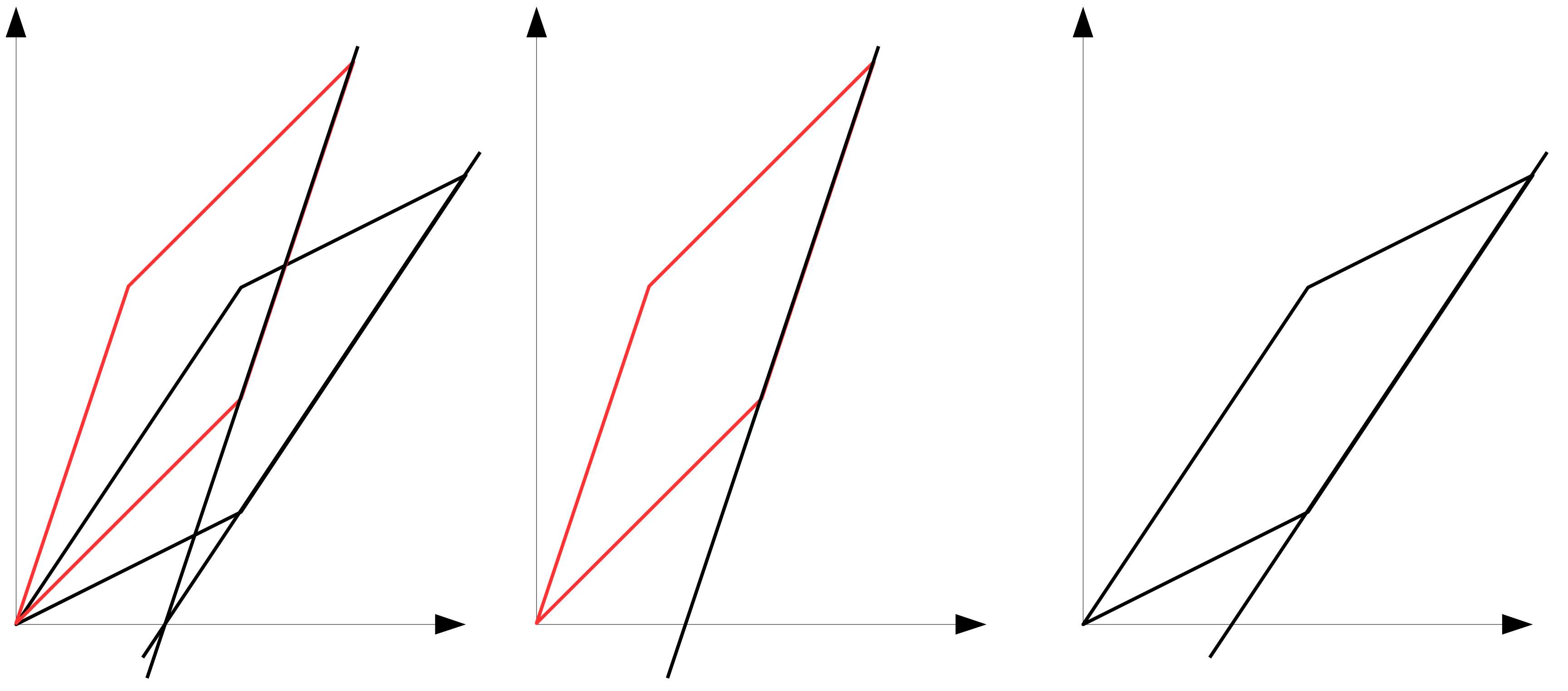

- Now we shear the parallelogram
  - Note: parallelogram area doesn't change if the shear follows the parallel line

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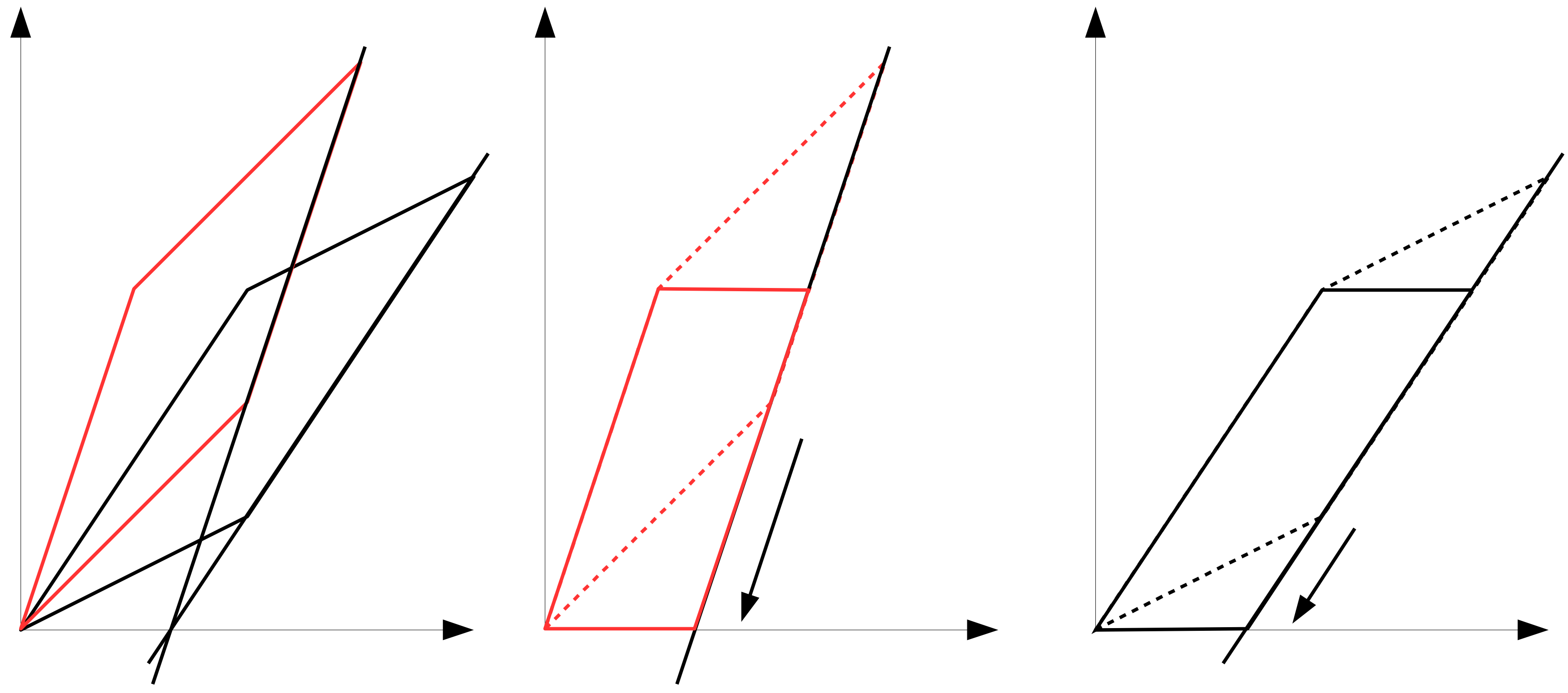

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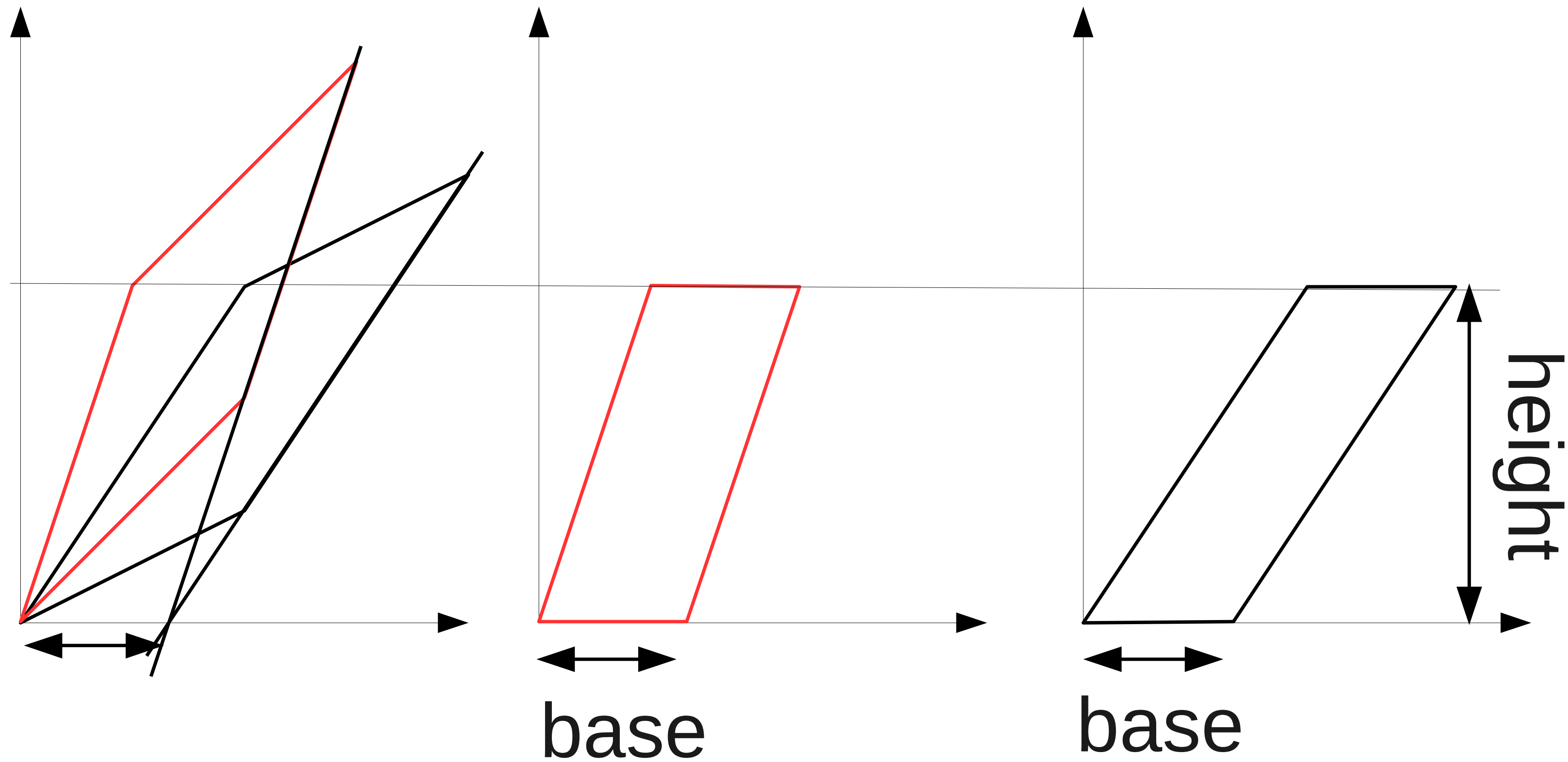
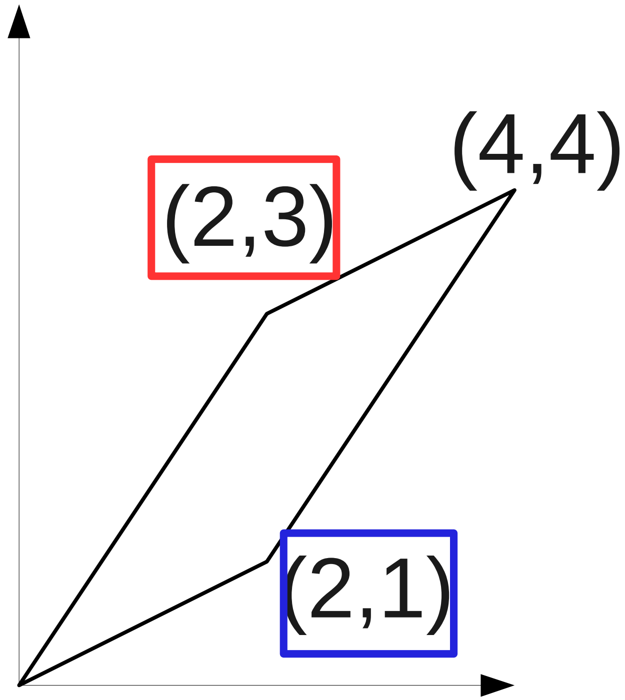


- Base and height are the same
  - The same area!

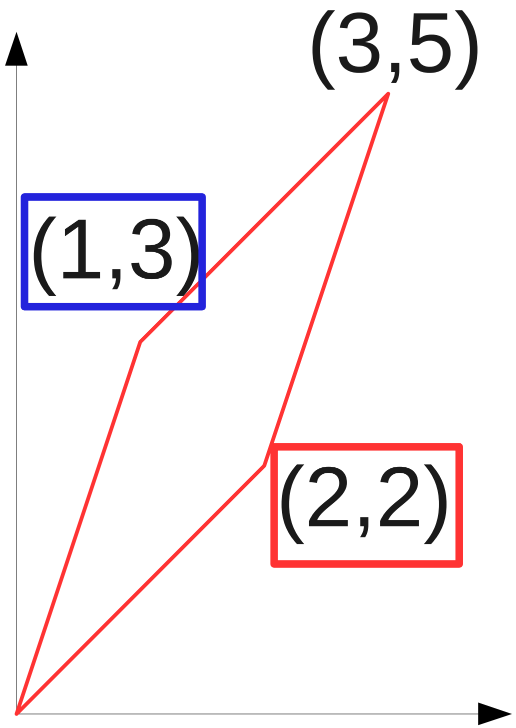
Why these two parallelogram has the same area (2)?



$$
\begin{aligned}
\left|\det\begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}\right| &= |2*1-3*2| \\
&= |2-6| \\
&= 4
\end{aligned}
$$

- Of course you can compute the determinant.
  - Area = 4

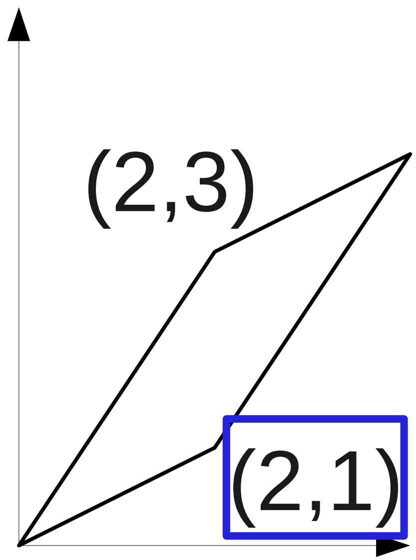
Why these two parallelogram has the same area (2)?



$$\left|\det\begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}\right| = |2*3-1*2| = |6-2| = 4$$

- Of course you can compute the determinant.
  - Area = 4
  - The same area! But this might not be so intuitive.

Elimination and shear again
The third explanation

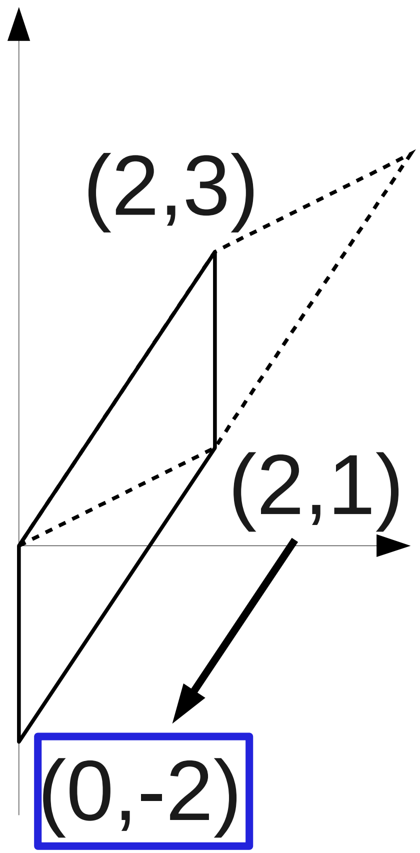


$$\begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}$$

row2 = row2 -row1

- Elimination doesn't change the determinant (= area)

Elimination and shear again
The third explanation



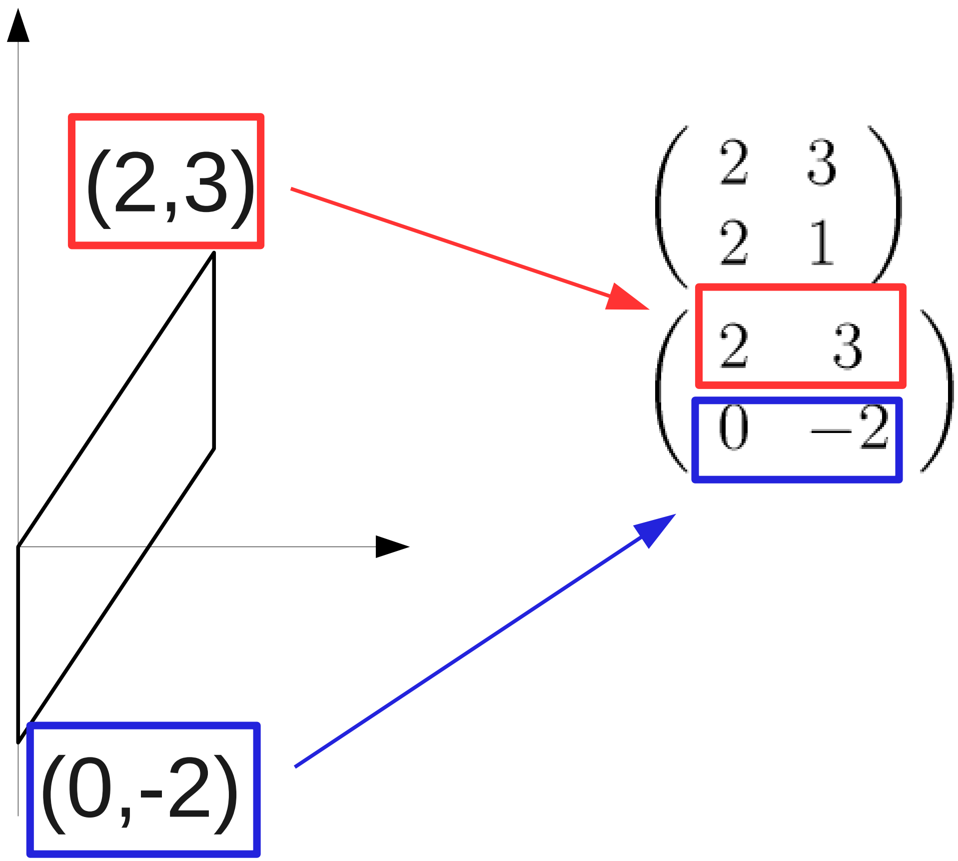
$$\begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}$$

row2 = row2 -row1

$$\begin{pmatrix} 2 & 3 \\ 0 & -2 \end{pmatrix}$$

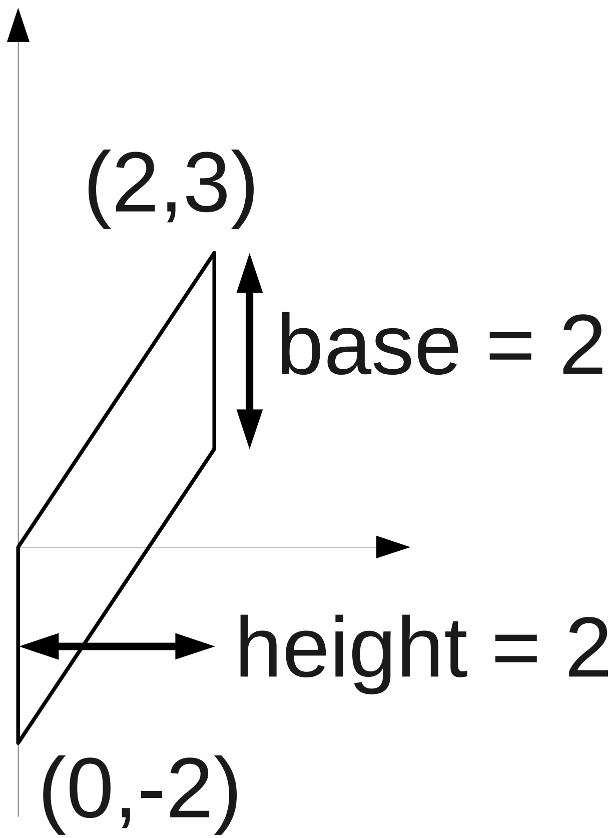
- Elimination doesn't change the determinant (= area)
- Shear the parallelogram

Elimination and shear again
The third explanation



- This is the new parallelogram
- See how the matrix tells the parallelogram

Elimination and shear again
The third explanation

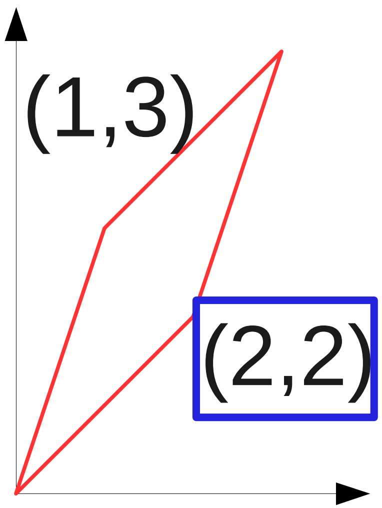


$$\begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 3 \\ 0 & -2 \end{pmatrix}$$

- Now the area is 4

Elimination and shear again
The third explanation



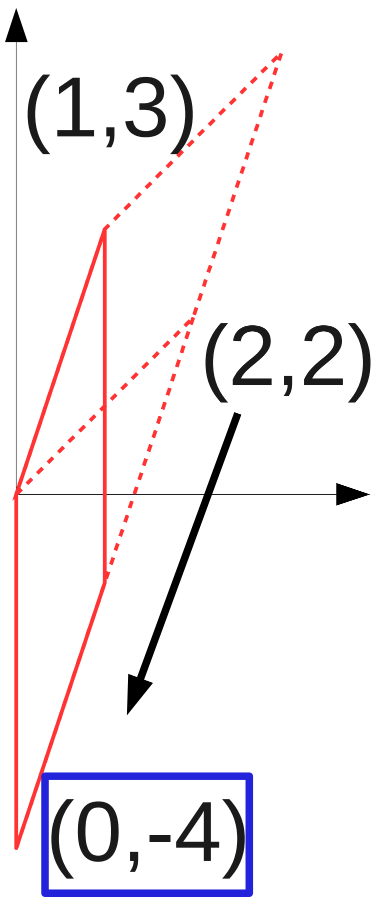
$$\begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}$$

row2 =

row2 - 2 * row1

- Do the same … elimination

Elimination and shear again
The third explanation



$$\begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 \\ 0 & -4 \end{pmatrix}$$

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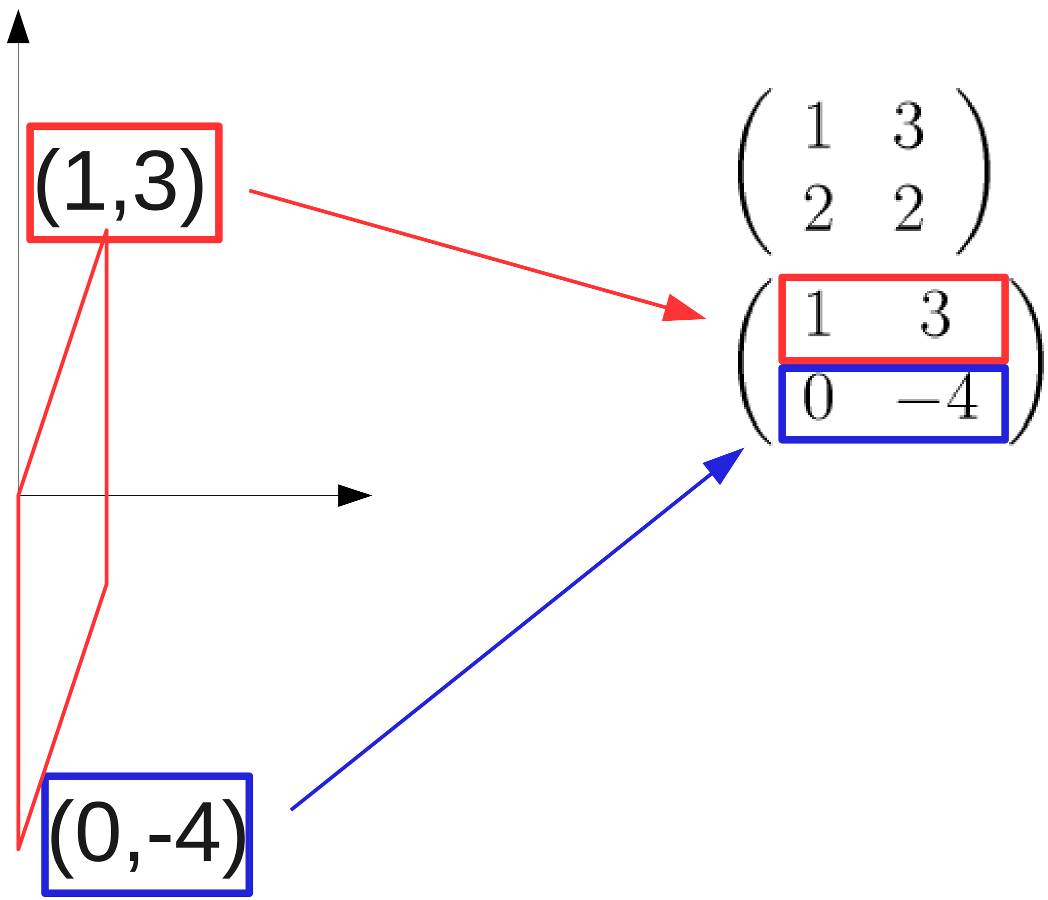
row2 - 2 * row1

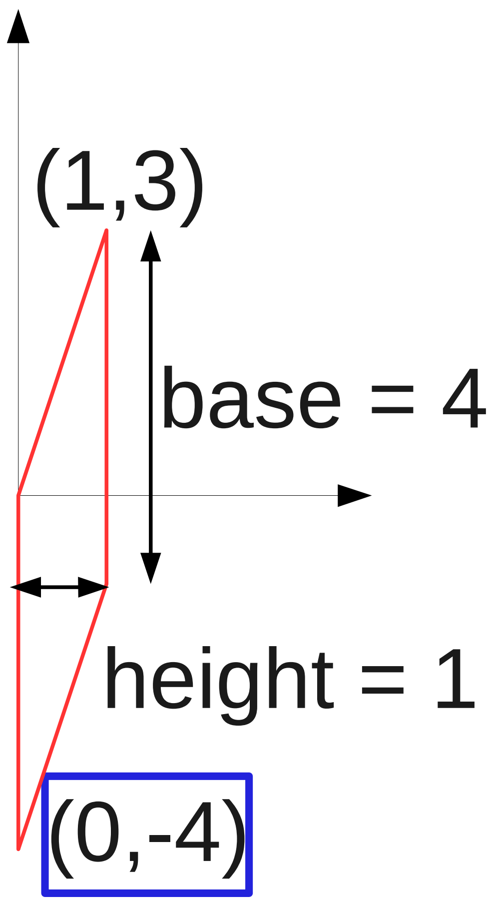
- Do the same … elimination

Elimination and shear again
The third explanation



- Here is the new matrix

Elimination and shear again
The third explanation



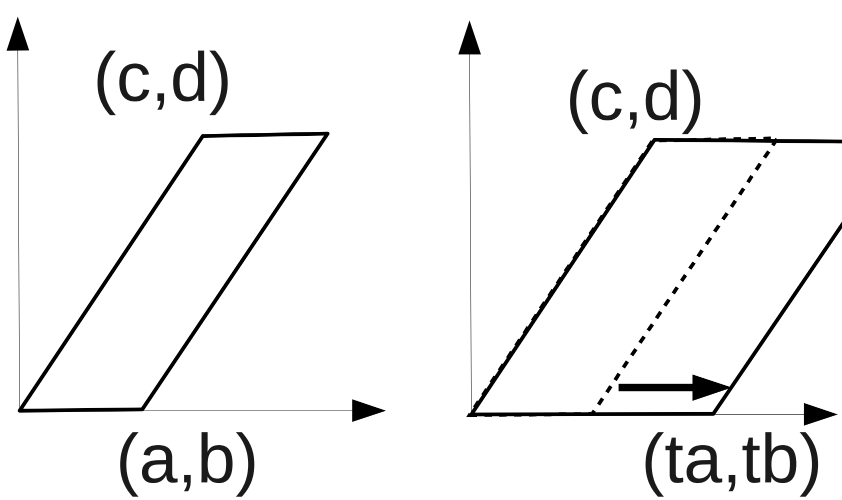
$$\begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 \\ 0 & -4 \end{pmatrix}$$

- The area is 4

Determinant's linearity

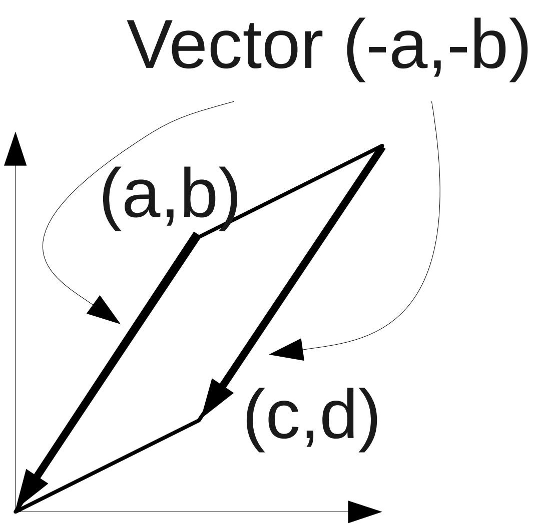
- The determinant is a linear function of each row separately (e.g., [1] p.246, property 3)
  - Multiply row1 by any number *t*
    - determinant is *t* times larger = area is *t* times larger

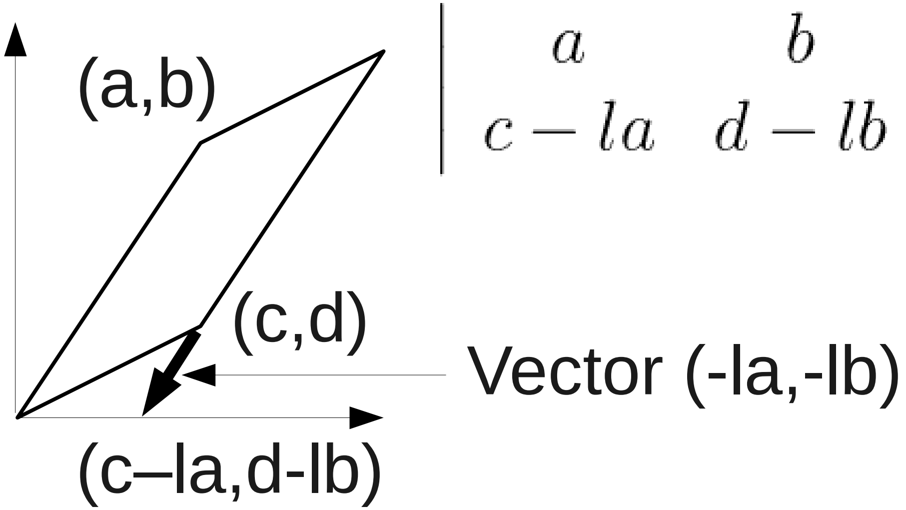


$$\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

Determinant's linearity

- The determinant is a linear function of each row separately (e.g., [1] p.246, property 5)
  - We saw a shear is elimination
    - *l* times row 1 from row 2 … no change the area/det





$$\begin{vmatrix} a & b \\ c - la & d - lb \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

Determinant's linearity

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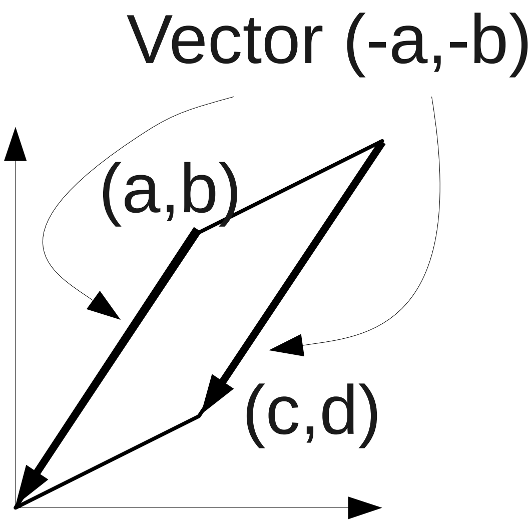
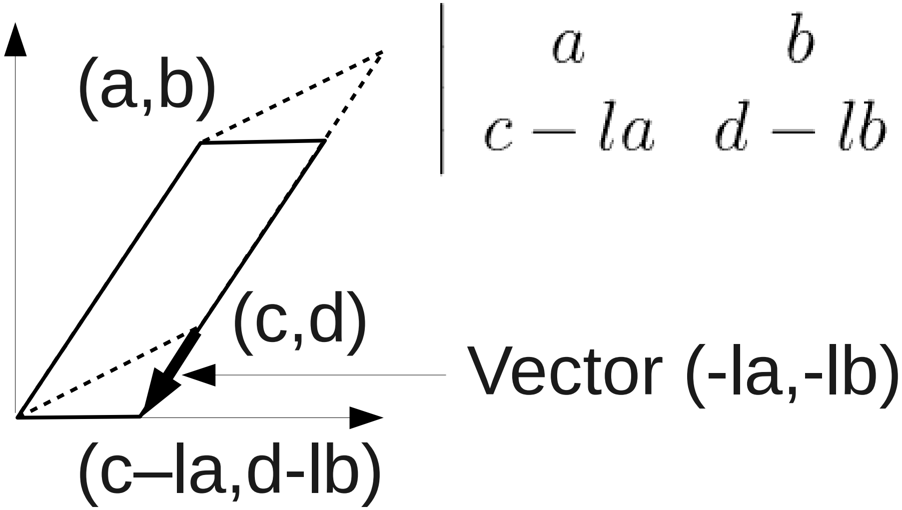
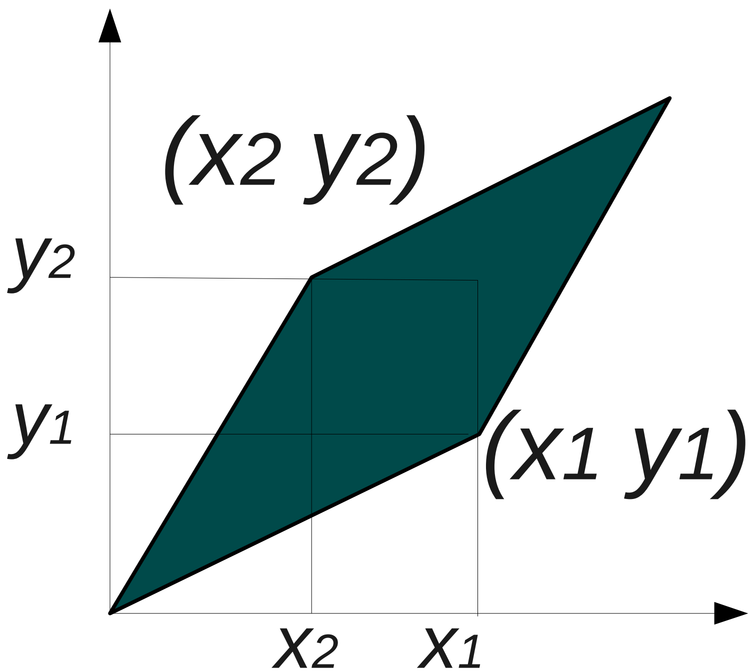




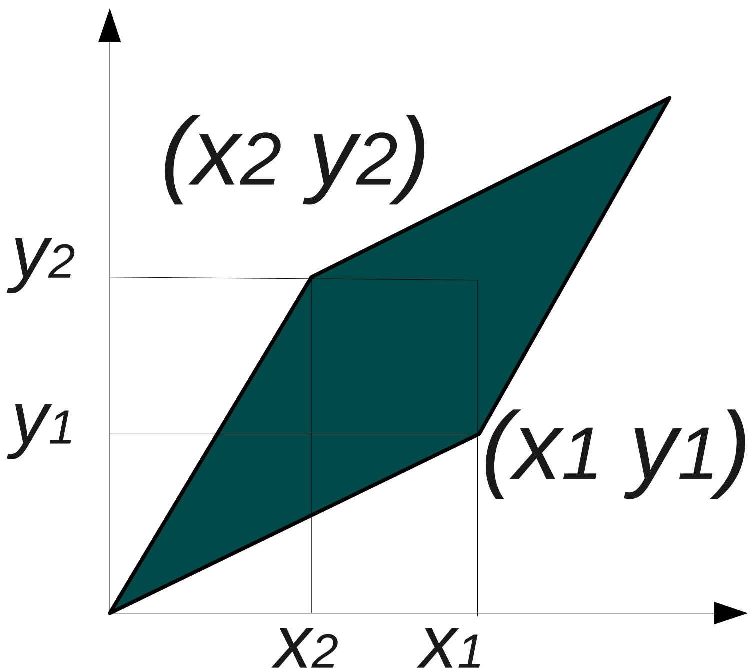
$$\begin{vmatrix} a & b \\ c - la & d - lb \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

Conclusion: area of parallelogram



- $|x1\ y2 - x2\ y1|$
- = determinant
- Parallelogram doesn't change the area when shear
- It's same to the elimination
  - Determinant doesn't change when elimination is performed (= sheared)

And more...



- There are many related topics
  - Cross product … k-forms
  - Jacobian
  - Stokes' theorem
  - First fundamental form
  - …
- All related with parallelogram/parallelopipe

References

- [1] Gilbert Strang, Introduction to Linear Algebra 4th ed.
- [2] Gerald Farin & Dianne Hansford, Practical Linear Algebra, A Geometry Toolbox
- [3] Thomas A. Garrity, All the Mathematics You Missed: But Need to Know for Graduate School
- ...